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Critical behaviour at an edge

John L Cardy
Department of Physics, University of California, Santa Barbara, California 93106, USA

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Abstract. The critical behaviour of a magnetic system with O(N) spin symmetry, bounded by two $(d-1)$-dimensional hyperplanes meeting at an angle α, is studied within mean field theory and in $d=4-\varepsilon$ dimensions. New exponents emerge for correlation functions, and magnetisation and susceptibilities, for spins close to the edge. They can be expressed in terms of known bulk and surface exponents, together with a single new edge exponent, which depends, however, on the angle α. This exponent is computed to first order in ε.

1. Introduction

Consider a homogeneous magnetic system in three dimensions, bounded by planar surfaces. In the thermodynamic limit, the total free energy has the decomposition

$$F = Vf_b + Af_s + Lf_e + \dots \tag{1.1}$$

where V is the volume, A is the total surface area, L is the total length of the edges, and so on. The bulk free energy per unit volume, f_b, depends on bulk quantities like the temperature and the applied magnetic field. The surface free energy per unit area, f_s, depends on these quantities, plus any local quantities, such as a magnetic field applied at the surface. Likewise, the edge free energy per unit length, f_e, depends on all the bulk and surface fields together with any fields localised near the edge. In addition, as we shall show, f_e depends on the angle α between the two planes defining the edge. At the bulk critical point, $f_b, f_s, f_e, \dots$ all exhibit singularities. The critical behaviour of surface quantities has been analysed extensively (Binder 1983). The various new exponents which arise are related by scaling laws to the bulk exponents, and one new independent surface exponent, which is determined by the renormalisation group (RG) eigenvalue of the surface magnetic field. In addition, the critical exponents determining the behaviour at criticality of the correlation functions between spins at the surface and in the bulk are related by scaling laws to surface and bulk thermodynamic exponents.

In this paper we study the critical behaviour of the edge free energy f_e, and of correlation functions involving spins near the edge, within the framework of mean field theory, and the renormalisation group and ε expansion. Since the edge free energy depends on bulk, surface and edge quantities, one can define a whole new genus of critical exponents describing the singular dependence on these quantities. However, the renormalisation group predicts scaling laws which relate all these new exponents to four basic ones. These are the eigenvalues of temperature, bulk magnetic field, surface field, and a new edge field. (Strictly speaking, the scaling fields are linear combinations of these fields with others of the same symmetry and extensiveness.)

The edge exponent turns out to be less universal than those of the bulk and the surface. For an isotropic system, it depends only on the angle α. This dependence is apparent even within mean field theory, where, for example, the correlation function between a spin close to the edge and one in the bulk decays as $r^{-(d-2+\eta_2)}$ at criticality, where $\eta_2=\pi/\alpha$. For spatially anisotropic systems, a non-universal dependence on the anisotropy also appears.

The outline of this paper is as follows. In § 2 we define the Landau–Ginzburg N-vector model within the restricted geometry, and discuss the correlation functions within mean field theory, which is applicable for dimension $d>4$. In § 3 the one-loop correction to the correlation function is evaluated, from which the critical exponent η_2 may be determined to $O(\varepsilon)$. The next section contains a complete RG analysis, where it is argued that one new edge operator renormalisation constant is required to render the theory finite. From this follows the full RG equation, and the scaling laws for the edge free energy and the correlation functions. We conclude with a summary of our results, and a discussion of further questions and potential applications.

2. Mean field theory

In order to simplify the analysis, we consider a wedge-shaped geometry bounded by two semi-infinite planes meeting at an angle α. We adopt cylindrical polar coordinates (ρ, θ, z) where z is, in general, a $(d-2)$-dimensional vector parallel to the edge, and $0\leqslant\theta\leqslant\alpha$. We shall refer to $\rho=0$ as an edge, even though for $d=2$, for example, it is a zero-dimensional corner. We consider a scalar field $\varphi(r)$ with N spin components, and the free energy functional

$$F\{\varphi\}=\int_V \mathrm{d}^d r[\tfrac{1}{2}(\nabla\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2+\tfrac{1}{4}u_0(\varphi^2)^2]+\int_S \mathrm{d}^{d-1}r(\tfrac{1}{2}c\varphi^2) \tag{2.1}$$

where the second term is a surface term, representing the weakening (for $c>0$) of the interaction strength at the surface. For the semi-infinite system Lubensky and Rubin (1975a, b) have analysed (2.1) within mean field theory and the renormalisation group. For $c>0$ (the ordinary transition) the surface orders at the same temperature as the bulk. Under the RG, c is driven to a fixed point at $+\infty$. In what follows, we shall consider only the ordinary transition, and so shall take $c=+\infty$. This forces the Dirichlet boundary condition $\varphi=0$ on S, which makes the analysis tractable.

Within mean field theory, the spin–spin correlation function is proportional to the bare propagator G_0 of the field theory defined by (2.1), satisfying

$$(-\nabla^2+m_0^2)G_0(r,r')=\delta^{(d)}(r-r') \tag{2.2}$$

with $m_0^2=\xi^{-2}\propto|T-T_c|$, and G_0 satisfying Dirichlet boundary conditions. Taking advantage of translation invariance in the $d-2$ transverse directions, we may write

$$G_0(r,r')=\int\frac{\mathrm{d}^{d-2}k}{(2\pi)^{d-2}}g_0(k;\rho,\theta;\rho',\theta')\exp[\mathrm{i}k\cdot(z-z')] \tag{2.3}$$

where

$$\left(-\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+k^2+\xi^{-2}\right)g_0=\frac{1}{\rho}\delta(\rho-\rho')\delta(\theta-\theta'). \tag{2.4}$$

The Green function g_0 may be expressed in two standard forms: as an eigenfunction expansion

$$g_0(\boldsymbol{k};\rho,\theta;\rho',\theta')=\frac{2}{\alpha}\sum_{n=1}^{\infty}\int_0^{\infty}\mu\,\mathrm{d}\mu\,\frac{J_{n\lambda}(\mu\rho)J_{n\lambda}(\mu\rho')}{\kappa^2+\mu^2}\sin n\lambda\theta\sin n\lambda\theta' \tag{2.5}$$

or more compactly as

$$g_0(\boldsymbol{k};\rho,\theta;\rho',\theta')=\frac{2}{\alpha}\sum_{n=1}^{\infty}I_{n\lambda}(\kappa\rho_<)K_{n\lambda}(\kappa\rho_>)\sin n\lambda\theta\sin n\lambda\theta' \tag{2.6}$$

where we have introduced the notation $\lambda=\pi/\alpha$, $\kappa^2=\boldsymbol{k}^2+\xi^{-2}$. As usual, $\rho_>$ ($\rho_<$) is the greater (lesser) of ρ and ρ'. From (2.5)–(2.6) we may infer the asymptotic behaviour of G_0 at the critical point $\xi^{-1}=0$, in various limits:

(a) $z=z'$, ρ fixed, $\rho'\to\infty$. Using (2.6) and rescaling $|\boldsymbol{k}|=x/\rho'$,

$$G_0\sim\frac{2}{\alpha}K_{d-2}\rho'^{-d+2}\int_0^{\infty}x^{d-3}\,\mathrm{d}x\sum_{n=1}^{\infty}I_{n\lambda}(x\rho/\rho')K_{n\lambda}(x)\sin n\lambda\theta\sin n\lambda\theta' \tag{2.7}$$

where $K_d=2\pi^{d/2}/\Gamma(d/2)(2\pi)^d$. In the limit $\rho/\rho'\to0$ we may approximate $I_{n\lambda}$ by its form for small argument. We then see that only $n=1$ is important in the sum, and

$$G_0\sim(A(\lambda)\rho^{\lambda}/\rho'^{d-2+\lambda})\sin\lambda\theta\sin\lambda\theta'. \tag{2.8}$$

This leads us to identify the exponent $\eta_2=\lambda=\pi/\alpha$. The amplitude

$$A(\lambda)=\frac{(2/\alpha)K_{d-2}}{2^{\lambda}\Gamma(\lambda+1)}\int_0^{\infty}\mathrm{d}x\,x^{d-3+\lambda}K_{\lambda}(x)\propto\frac{\lambda\Gamma(\lambda+1+d/2)}{2^{\lambda}\Gamma(\lambda+1)}. \tag{2.9}$$

Although the exponent η_2 becomes arbitrarily large as $\alpha\to0$, the amplitude diverges in that limit. The limits $\alpha\to0$ and $\rho'/\rho\to\infty$ do not commute.

(b) ρ,ρ' fixed, $|z-z'|\equiv r\to\infty$. In this case form (2.5) is more appropriate, rescaling $\boldsymbol{k}=\bar{\boldsymbol{k}}/r$, $\mu=\bar{\mu}/r$. Once again only $n=1$ is important, and the result is proportional to

$$[(\rho\rho')^{\lambda}/r^{d-2+2\lambda}]\sin\lambda\theta\sin\lambda\theta' \tag{2.10}$$

from which we identify the exponent $\eta_{2,2}=2\pi/\alpha$.

Other cases may be dealt with similarly. We establish a notation as follows: a subscript 2 will refer to edge quantities, and 1 to surface quantities. A subscript 0 is implied for bulk quantities, but is not written explicitly. The critical exponent describing the critical decay of a correlation function for spins in regions p and q is then written $\eta_{p,q}$. In this notation, the surface exponents (Binder and Hohenberg 1972) are $\eta_{\perp}=\eta_1$ and $\eta_{\|}=\eta_{1,1}$. The mean field results are summarised by

$$\eta_{p,q}=\tfrac{1}{2}(\eta_{p,p}+\eta_{q,q}) \tag{2.11}$$

$$\begin{aligned}&\eta_{0,0}=\eta=0 &&\text{(bulk)}\\ &\eta_{1,1}=\eta_{\|}=2 &&\text{(surface)}\\ &\eta_{2,2}=2\pi/\alpha &&\text{(edge).}\end{aligned} \tag{2.12}$$

Note that the edge exponent agrees with the surface exponent for $\alpha=\pi$, as it must. The scaling relations (2.11) will be seen to survive below four dimensions (§ 4). The relation $\eta_{\|}=2\eta_{\perp}-\eta$ (Lubensky and Rubin 1975a) is a special case.

We now give examples of the mean field calculation of the edge magnetisation and susceptibility exponents. First consider the effect of adding a small bulk magnetic field, h, for $T > T_c$. The local magnetisation $\varphi(\boldsymbol{r})$ will satisfy

$$-\nabla^2\varphi + \xi^{-2}\varphi = h \tag{2.13}$$

so that

$$\varphi(\boldsymbol{r}) = h\int \mathrm{d}^d r'\, G_0(\boldsymbol{r}, \boldsymbol{r}') \tag{2.14}$$

$$= h(2/\alpha)\sum_n \int \rho'\,\mathrm{d}\rho'\,\mathrm{d}\theta'\, I_{n\lambda}(\rho_</\xi)K_{n\lambda}(\rho_>/\xi)\sin n\lambda\theta \sin n\lambda\theta'. \tag{2.15}$$

The dominant contribution comes from $\rho' > \rho$. The ρ' integration then gives a factor ξ^2, and for $\rho \to 0$ we obtain an additional factor $(\rho/\xi)^\lambda$. Hence the edge susceptibility $\partial\varphi_e/\partial h|_{h=0}$ behaves like $|T - T_c|^{-\gamma_2}$ with $\gamma_2 = 1 - \lambda/2 = 1 - \pi/2\alpha$.

For $T < T_c$, with $h = 0$, the local magnetisation must be obtained from the nonlinear equation

$$-\nabla^2\varphi - \xi^{-2}\varphi + u_0\varphi^3 = 0 \tag{2.16}$$

with the boundary condition that $\varphi \to \xi^{-1}u_0^{-1/2}$ (the bulk value) as $\rho \to \infty$. Although (2.13) cannot be solved by simple quadrature (as in the surface case (Binder and Hohenberg 1972)), its solution must have the scaling form

$$\varphi(\boldsymbol{r}) = \xi^{-1}u_0^{-1/2}f(\rho/\xi, \theta). \tag{2.17}$$

As $\rho \to 0$ at fixed ξ, we can ignore the nonlinear term, so that φ must behave like ρ^λ. We conclude that for fixed ρ and $\xi \to \infty$, $\varphi(\boldsymbol{r}) \propto \xi^{-\lambda-1} \propto (T_c - T)^{1/2+\lambda/2}$. This gives the edge magnetisation exponent $\beta_2 = \frac{1}{2} + \pi/2\alpha$.

Rather than derive the whole list of mean field exponents at this stage, we refer the reader to the scaling laws of § 4, where the mean field results may be obtained by setting $d = 4$.

3. One-loop calculation

In this section we evaluate the full correlation function $G(r, \boldsymbol{r}')$ to first order in u. We shall perform the calculation in the massless theory ($T = T_c$), close to $d = 4$. The Feynman diagrams are shown in figure 1. Since the transverse momentum $\boldsymbol{k}$ is conserved we may decompose the full G as in (2.3). Then, to one loop,

$$g(\boldsymbol{k}; \rho, \theta; \rho', \theta') = g_0(\boldsymbol{k}; \rho, \theta; \rho', \theta') + g_1(\boldsymbol{k}; \rho, \theta; \rho', \theta') \tag{3.1}$$

where

$$g_1 = -(N+2)u_0 \int_0^\infty \rho''\,\mathrm{d}\rho'' \int_0^\alpha \mathrm{d}\theta''\, g_0(\boldsymbol{k}; \rho, \theta; \rho'', \theta'')g_0(\boldsymbol{k}; \rho'', \theta''; \rho', \theta')$$
$$\times \int \frac{\mathrm{d}^{d-2}k'}{(2\pi)^{d-2}} g_0(\boldsymbol{k}'; \rho'', \theta''; \rho'', \theta''). \tag{3.2}$$

When the expressions (2.5) or (2.6) are substituted for g_0, the arguments of § 2 show that only the term proportional to $\sin\lambda\theta$ is important if ρ is small. Likewise, it will be shown later that we need consider only the term proportional to $\sin\lambda\theta'$. Within

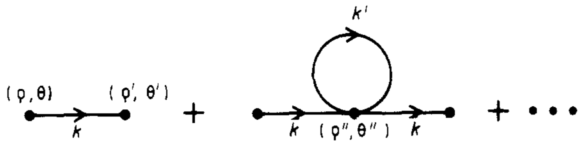

Figure 1. Diagrams contributing to $G(\boldsymbol{r}, \boldsymbol{r}')$ at one-loop level.

the loop, however, all harmonics must be summed over. The θ'' integration then involves

$$\frac{2}{\alpha}\int_0^\alpha \mathrm{d}\theta'' \sin^2\lambda\theta'' \sum_n I_{n\lambda}(k'\rho'')K_{n\lambda}(k'\rho'')\sin^2 n\lambda\theta''$$

$$=\tfrac{1}{4}\sum_{n=1}^{\infty}(2+\delta_{n,1})I_{n\lambda}(k'\rho'')K_{n\lambda}(k'\rho''). \tag{3.3}$$

The next step is to carry out the k' integration. This has the form

$$\int \mathrm{d}^{d-2}k'\, I_{n\lambda}(k'\rho'')K_{n\lambda}(k'\rho'') = K_{d-2}\rho''^{-d+2}\int_0^\infty x^{d-3}\,\mathrm{d}x\, I_{n\lambda}(x)K_{n\lambda}(x) \tag{3.4}$$

which may be written (Abramowitz and Stegun 1965, Gradshteyn and Ryzhik 1980)

$$K_{d-2}\rho''^{-d+2}\frac{\Gamma(d/2-1+n\lambda)\Gamma(d/2-1)}{2^{4-d}\Gamma(n\lambda+1)}\,{}_2F_1(d/2-1+n\lambda, d/2-1; n\lambda+1; 1) \tag{3.5}$$

$$=K_{d-2}\rho''^{-d+2}\frac{\Gamma(d/2-1)\Gamma(d/2-1+n\lambda)\Gamma(3-d)}{2^{4-d}\Gamma(2-d/2)\Gamma(2-d/2+n\lambda)} \tag{3.6}$$

$$=\frac{\rho''^{-d+2}}{2\pi^{d/2-1}}\frac{1}{\Gamma(2-d/2)}\int_0^1 \mathrm{d}t\, t^{d/2-2+n\lambda}(1-t)^{2-d}. \tag{3.7}$$

In this last form the sum over n implied in (3.3) may be carried out. Putting all these results together,

$$g_1 \sim -(N+2)u_0(2/\alpha)f(\lambda,d)\sin\lambda\theta\sin\lambda\theta'$$

$$\times\int_0^\infty \rho''^{3-d}\,\mathrm{d}\rho''\, I_\lambda(k\rho_<)K_\lambda(k\rho_>)I_\lambda(k\rho'_<)K_\lambda(k\rho'_>) \tag{3.8}$$

where $\rho_> = \max(\rho, \rho'')$ and $\rho'_> = \max(\rho', \rho'')$ and

$$f(\lambda,d)=\frac{(2/\alpha)}{8\pi^{d/2-1}\Gamma(2-d/2)}\int_0^1 \mathrm{d}t(1-t)^{2-d}t^{d/2-2+\lambda}\left(\frac{2}{1-t^\lambda}+1\right). \tag{3.9}$$

$f(\lambda, d)$ has a simple pole at $d=2$. Its residue is independent of α, however, and it is cancelled by the bulk mass counterterm. The apparent pole at $d=3$ may be shown to have zero residue. This is an important consistency check, as there is no counterterm available to cancel it. The integral in (3.9) has a simple pole at $d=4$, and thus $f(\lambda, 4)$ is *finite*. After some algebra,

$$f(\lambda,4) = -(5\lambda^2+1)/48\pi^2. \tag{3.10}$$

The simplest way to extract the $O(\varepsilon)$ correction to the exponent η_2 is to take u_0 equal to its fixed point value of $8\pi^2\varepsilon/(N+8)$ (Wilson and Fisher 1972), evaluate

(3.8) at $d=4$, and assume that the logarithm exponentiates. A more systematic approach will be found in § 4. The contribution to (3.8) from the region $\rho \leqslant \rho'' \leqslant \rho'$ is proportional to

$$I_\lambda(k\rho)K_\lambda(k\rho')\int_\rho^{\rho'} \frac{d\rho''}{\rho''} I_\lambda(k\rho'')K_\lambda(k\rho''). \tag{3.11}$$

Since, as $\rho'' \to \infty$, $I_\lambda(k\rho'')K_\lambda(k\rho'') \propto 1/\rho''$, but as $\rho'' \to 0$, it goes to a constant $(1/2\lambda)$, we see that the logarithm comes from the region $\rho'' \gg \rho$. This is in accord with the idea that, to one loop, there should be no contribution to η_2 from the bulk region $\rho'' \sim \rho'$, and we are dealing with an edge renormalisation effect. One can check that no logarithm comes from the regions $\rho'' < \rho$ or $\rho'' > \rho'$. Also, if we had kept terms proportional to $\sin n'\lambda\theta'$ with $n' > 1$, the integral would be proportional to $\int d\rho''\, \rho''^{-1} I_{n'\lambda}(k\rho'')K_\lambda(k\rho'')$, and so would once again generate no logarithm.

We conclude that

$$g_1 \sim -(N+2)u_0(2/\alpha)f(\lambda, 4) \sin\lambda\theta \sin\lambda\theta' \times (1/2\lambda)I_\lambda(k\rho)K_\lambda(k\rho')[-\ln(k\rho) + \text{finite terms as } \rho \to 0]. \tag{3.12}$$

Putting this together with g_0, and evaluating $G(\boldsymbol{r}, \boldsymbol{r}')$ as $\rho \to 0$ by rescaling $|\boldsymbol{k}|$ as in (2.7),

$$G \sim \frac{2}{\alpha} \frac{K_{d-2}\rho'^{-d+2}}{2\Gamma(1+\lambda)} \sin\lambda\theta \sin\lambda\theta' \int_0^\infty x^{d-3}\, dx\, (x\rho/\rho')^\lambda K_\lambda(x) \times \{1 + [(N+2)u_0 f(\lambda, 4)/2\lambda] \ln(x\rho/\rho') + \ldots\}. \tag{3.13}$$

With u_0 equal to its fixed point value, assuming exponentiation of the logarithm in (3.13), we find $G \propto \rho'^{-d+2}(\rho/\rho')^{\eta_2}$, with

$$\eta_2 = \lambda + \frac{8\pi^2 f(\lambda, 4)(N+2)\varepsilon}{2\lambda(N+8)} = \lambda - \frac{\varepsilon(5\lambda^2+1)(N+2)}{12\lambda(N+8)}. \tag{3.14}$$

Note that this agrees with the $O(\varepsilon)$ value for $\eta_\perp$ found by Lubensky and Rubin (1975a), when $\lambda = 1$ ($\alpha = \pi$). The $O(\varepsilon)$ term in η_2 also grows linearly with λ as $\lambda \to \infty$. From (3.13) the amplitude is now proportional to $\int_0^\infty x^{d-3+\eta_2} K_\lambda(x)\, dx$, and this is only convergent for

$$d - 2 + \eta_2 > \lambda. \tag{3.15}$$

At fixed small ε this inequality is violated for sufficiently large λ. Using (3.14), the amplitude will diverge for

$$\alpha/\pi < 5(N+2)\varepsilon/24(N+8) + O(\varepsilon^2) \tag{3.16}$$

which corresponds to $\alpha \approx 12°$ for $N=1$, $d=3$. Without taking our calculation to higher orders, it is not clear whether this signals new behaviour, or is an artifact of the ε expansion.

4. Renormalisation of edge operators and scaling laws

Within the context of a field theoretical RG calculation, the appearance of new exponents should be connected with the appearance of new ultraviolet divergences in quantities evaluated near the edge. In this section we make this explicit. A similar argument for the surface case has been given by Diehl and Dietrich (1981) (see also Symanzik (1981)).

Consider a magnetic field coupling to spins close to the edge. This field is localised over a distance much smaller than the bulk correlation length. For general values of c in (2.1) the magnetisation close to the edge will have the expansion

$$\varphi(z, \rho, \theta) = \int \frac{d^{d-2}\boldsymbol{k}}{(2\pi)^{d-2}} e^{i\boldsymbol{k}\cdot\boldsymbol{z}} \sum_{n=0}^{\infty} I_{n\lambda}(k\rho)[A^{(n)}(\boldsymbol{k}) \sin n\lambda\theta + B^{(n)}(\boldsymbol{k}) \cos n\lambda\theta] \tag{4.1}$$

and the edge magnetic field will couple to all terms in this sum. For finite c the dominant term as $\rho \to 0$ is that proportional to $B^{(0)}(\boldsymbol{k})$. However, at the fixed point, the condition $c \to \infty$ forces all the $B^{(n)}$ to vanish, so the leading term is $A^{(1)}$. Reverting to position space, we therefore define the edge operators

$$A^{(n)}(z) = \lim_{\rho \to 0} \frac{2\rho^{-n\lambda}}{\alpha} \int_0^{\alpha} d\theta\, \varphi(z, \rho, \theta) \sin n\lambda\theta. \tag{4.2}$$

$A^{(n)}$ has canonical dimension $d/2 - 1 + n\lambda$. The most relevant operator will therefore be $A^{(1)}$ (Amit 1978), which may also be written

$$A^{(1)}(z) = \lim_{\rho \to 0} \frac{\varphi(z, \rho, \theta)}{\rho^{\lambda} \sin \lambda\theta}. \tag{4.3}$$

For $\lambda = 1$, $A^{(1)}$ coincides with the surface normal derivative operator considered by Diehl and Dietrich (1981).

In the infinite system, the theory at criticality is rendered finite by two renormalisation constants

$$\varphi = Z_{\varphi}^{1/2} \varphi_{R} \tag{4.4}$$

$$g = \kappa^{\varepsilon} Z_u u_0 \tag{4.5}$$

where g is the dimensionless renormalised coupling constant, and κ is an arbitrary momentum scale. The renormalised Green functions

$$G^{(N)R} = Z_{\varphi}^{-N/2} G^{(N)} = Z_{\varphi}^{-N/2} \langle \varphi(r_1) \ldots \varphi(r_N) \rangle_c \tag{4.6}$$

are then finite when expressed in terms of g.

If we do not add edge or surface magnetic fields, the same renormalisation will render $G^{(N)R}$ finite in the wedge geometry also. This is because any potential new divergences from integrations near the boundary are suppressed because $\varphi = 0$ there.

To discuss $T \neq T_c$, we add a term $t \int_V \varphi(r)^2 d^d r$ to the Hamiltonian, and expand $G^{(N)}$ in powers of t. This amounts to considering the $G^{(N)}$ with an arbitrary number M of insertions of $\varphi(r)^2$, denoted by $G^{(N,M)}$. This is rendered finite by introducing one more renormalisation constant Z_{φ^2} so that

$$G^{(N,M)R} = Z_{\varphi}^{-N/2} Z_{\varphi^2}^{M} G^{(N,M)} \tag{4.7}$$

is finite. From the arbitrariness in the scale κ, $G^{(N,M)R}$ satisfies the RG equation (Amit 1978)

$$[\kappa\, \partial/\partial\kappa + \beta(g)\, \partial/\partial g + \tfrac{1}{2} N\eta(g) + M(2 - \nu^{-1}(g))] G^{(N,M)R} = 0 \tag{4.8}$$

where

$$\beta(g) = \kappa\, \partial g/\partial\kappa \tag{4.9}$$

$$\eta(g) = \kappa\, \partial \ln Z_{\varphi}/\partial\kappa \tag{4.10}$$

$$2-\nu^{-1}(g) = -\kappa\, \partial \ln Z_{\varphi^2}/\partial\kappa \tag{4.11}$$

all derivatives being taken at fixed bare coupling. At the fixed point $g = g^*$, for $t \neq 0$,

$$[\kappa\, \partial/\partial\kappa + \tfrac{1}{2}N\eta + (2-\nu^{-1})t\, \partial/\partial t]G^{(N)\mathrm{R}}(t) = 0 \tag{4.12}$$

where η, ν now refer to fixed point values. $G^{(N)}$ has canonical dimension $\frac{1}{2}N(d-2)$, so that

$$[-l\, \partial/\partial l + \kappa\, \partial/\partial\kappa + 2t\, \partial/\partial t - \tfrac{1}{2}N(d-2)]G^{(N)}(lr_i, t) = 0 \tag{4.13}$$

where we have used the fact that the canonical dimension of t is 2. Subtracting (4.13) from (4.12)

$$[l\, \partial/\partial l + \tfrac{1}{2}N(d-2+\eta) - \nu^{-1}t\, \partial/\partial t]G^{(N)\mathrm{R}}(lr_i, t) = 0. \tag{4.14}$$

All this is as in the infinite system. In the wedge geometry, however, we can conclude from (4.14) only that

$$G^{(2)}(\rho, \theta, z; \rho', \theta', z'; t) = \kappa^{d-2}(\kappa\rho')^{-(d-2+\eta)}\Psi\left(\frac{|z-z'|}{\rho'}, \frac{\rho}{\rho'}, \frac{\rho'}{\xi}, \theta, \theta'\right) \tag{4.15}$$

where $\xi \propto t^{-\nu}$ and Ψ is a scaling function whose behaviour, as $\rho/\rho' \to 0$, for example, is so far undetermined.

In order to complete the analysis we must therefore understand the renormalisation of edge operators. Following Diehl and Dietrich (1981), we add a term

$$h_2 \int \mathrm{d}^{d-2}z\, A^{(1)}(z) \tag{4.16}$$

to the Hamiltonian. We consider Green functions $G^{(N,M,L)}$ with M φ^2 insertions and L insertions of $A^{(1)}$. This, we assume, will be rendered finite by introducing a new edge renormalisation constant Z_2, so that

$$G^{(N,M,L)\mathrm{R}} = Z_{\varphi}^{-N/2} Z_{\varphi^2}^{M} Z_2^{L} G^{(N,M,L)} \tag{4.17}$$

This will satisfy

$$[\kappa\, \partial/\partial\kappa + \beta(g)\, \partial/\partial g + \tfrac{1}{2}N\eta(g) + M(2-\nu^{-1}(g)) + L(d/2-1-\lambda-y_2(g))]G^{(N,M,L)\mathrm{R}} = 0 \tag{4.18}$$

where

$$d/2-1-\lambda-y_2(g) = -\kappa\, \partial \ln Z_2/\partial\kappa. \tag{4.19}$$

Note that $d/2-1-\lambda$ is the canonical dimension of h_2. For t and h_2 non-zero, we then have, in analogy with (4.12),

$$[\kappa\, \partial/\partial\kappa + \tfrac{1}{2}N\eta + L(d/2-1-\lambda-y_2) + (2-\nu^{-1})t\, \partial/\partial t + (d/2-1-\lambda-y_2)h_2\, \partial/\partial h_2]G^{(N,0,L)\mathrm{R}}(t, h_2) = 0 \tag{4.20}$$

at the fixed point. Now $G^{(N,0,L)}$ has canonical dimension $\frac{1}{2}(N+L)(d-2) + L\lambda$. Thus, in analogy with (4.14)

$$[l\,\partial/\partial l+\tfrac{1}{2}N(d-2+\eta)+L(d-2-y_2)-\nu^{-1}t\,\partial/\partial t-y_2h_2\,\partial/\partial h_2] \times G^{(N,0,L)\mathrm{R}}(lr_i,t,h_2)=0. \qquad (4.21)$$

This equation is sufficient to determine all the exponents and the scaling laws between them. First, take $t=h_2=0$. The correlation function of two edge operators $G^{(0,0,2)}$ is

$$\langle A^{(1)}(z)A^{(1)}(z')\rangle_c \propto |z-z'|^{-(d-2+\eta_{2,2})} \qquad (4.22)$$

where

$$\eta_{2,2}=d-2-2y_2. \qquad (4.23)$$

The correlation function $G^{(1,0,1)}$ of an edge operator with the bulk is

$$\langle A^{(1)}(z)\varphi(z,\rho,\theta)\rangle \propto \rho^{-(d-2+\eta_2)} \qquad (4.24)$$

where

$$\eta_2=\tfrac{1}{2}(d-2)+\tfrac{1}{2}\eta-y_2. \qquad (4.25)$$

In order to compute correlations with surface spins, we may introduce a surface operator $A_\perp$ (Diehl and Dietrich 1981), and a corresponding index y_1 (which equals y_2+1, for $\lambda=1$). Then, for the correlation function of an edge operator with a surface operator,

$$\langle A^{(1)}(z)A_\perp(z,\rho,0)\rangle_c \propto \rho^{-(d-2+\eta_{1,2})} \qquad (4.26)$$

where

$$\eta_{1,2}=d-1-y_1-y_2. \qquad (4.27)$$

Similarly

$$\eta_\parallel=\eta_{1,1}=d-2y_1 \qquad (4.28)$$

$$\eta_\perp=\eta_1=\tfrac{1}{2}(d+\eta)-y_1. \qquad (4.29)$$

Note that the scaling relations (2.11) are automatically satisfied. The scaling part of the edge free energy will satisfy (4.20) with $N=L=0$. Since the edge free energy per unit length f_e has dimension $d-2$, the scaling part will satisfy

$$f_e^{(s)}=t^{(d-2)\nu}\psi(ht^{-y_0\nu},h_1t^{-y_1\nu},h_2t^{-y_2\nu}) \qquad (4.30)$$

where we have included the dependence on the bulk and surface magnetic fields, h and h_1 respectively. From this expression the exponents governing all possible magnetisations and susceptibilities may be read off. In particular, we note the edge magnetisation $\partial f_e/\partial h_2 \propto (-t)^{\beta_2}$ with $\beta_2=\nu(d-2-y_2)$, the edge susceptibility $\partial^2 f_e/\partial h_2\,\partial h \propto |t|^{-\gamma_2}$ with $\gamma_2=\nu(y_0+y_2-d+2)$, and the local susceptibility $\partial^2 f_e/\partial^2 h_2 \propto |t|^{-\gamma_{2,2}}$ where $\gamma_{2,2}=\nu(2y_2-d+2)$. The edge free energy has a singularity $|t|^{2-\alpha_e}$ with $\alpha_e=\alpha+2\nu$. By eliminating the eigenvalues y_i, many scaling relations may be derived.

The mean field results for the correlation exponents $\eta_{p,q}$ and the susceptibility exponents $\gamma_{p,q}$ are consistent with y_0, y_1, y_2 taking their values at the Gaussian fixed point, namely

$$y_0=d/2+1 \qquad y_1=d/2-1 \qquad y_2=d/2-1-\lambda. \qquad (4.31)$$

It remains to show how to compute the eigenvalue y_2 for $d=4-\varepsilon$, using its definition (4.19). The renormalisation constant Z_2 may be determined in the minimal subtraction scheme by demanding that it cancel the poles in ε of any one of the $G^{(N,M,L)}$. Rather than repeating the calculation of § 3, we shall instead consider $G^{(0,0,2)}$, and in particular its Fourier transform

$$g(\boldsymbol{k})=\int\frac{\mathrm{d}^{d-2}\boldsymbol{k}}{(2\pi)^{d-2}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{z}}\langle A^{(1)}(\boldsymbol{z})A^{(1)}(\boldsymbol{0})\rangle. \tag{4.32}$$

From (2.5), the zero-loop contribution to $g(\boldsymbol{k})$ is

$$g_0(\boldsymbol{k})=\frac{(2/\alpha)}{2^{2\lambda}\Gamma(1+\lambda)^2}\int_0^\infty\frac{\mu^{1+2\lambda}\,\mathrm{d}\mu}{\mu^2+\boldsymbol{k}^2}. \tag{4.33}$$

This integral is ultraviolet divergent. If we impose a cut-off $\mu<\Lambda$, the Λ-dependent terms correspond to short-distance singularities of $\langle A^{(1)}(\boldsymbol{z})A^{(1)}(\boldsymbol{0})\rangle$, which must be subtracted off, but in any case do not contribute to the $|\boldsymbol{z}|\to\infty$ behaviour. (An analogous situation arises for correlations of composite operators (Amit 1978).) The remaining finite term may be obtained by evaluating (4.33) for $\lambda<0$ and analytically continuing:

$$g_0(\boldsymbol{k})^{\text{sub}}=-(2/\alpha)\pi|\boldsymbol{k}|^{2\lambda}/2^{2\lambda+1}\Gamma(1+\lambda)^2\sin\pi\lambda. \tag{4.34}$$

The one-loop term is, from (2.6),

$$g_1(\boldsymbol{k})=\frac{-(N+2)u_0(2/\alpha)^2|\boldsymbol{k}|^{2\lambda}}{2^{2\lambda}\Gamma(1+\lambda)^2}\int_0^\infty\rho\,\mathrm{d}\rho\int_0^\alpha\mathrm{d}\theta\,K_\lambda(k\rho)^2\sin^2\lambda\theta$$
$$\times\int\frac{\mathrm{d}^{d-2}k'}{(2\pi)^{d-2}}g_0(\boldsymbol{k}';\rho,\theta;\rho,\theta). \tag{4.35}$$

The θ and k' integrations proceed as in § 3. The result is

$$g_1(\boldsymbol{k})=\frac{-(N+2)u_0(2/\alpha)|\boldsymbol{k}|^{2\lambda}}{2^{2\lambda}\Gamma(1+\lambda)^2}f(\lambda,d)\int_0^\infty\rho^{3-d}\,\mathrm{d}\rho\,K_\lambda(k\rho)^2. \tag{4.36}$$

The remaining integral gives (Gradshteyn and Ryzhik 1980)

$$2^{1-d}|\boldsymbol{k}|^{d-4}\Gamma(2-d/2)^2\Gamma(2-d/2+\lambda)\Gamma(2-d/2-\lambda)/\Gamma(4-d). \tag{4.37}$$

The poles at $d=4-2\lambda, 2-2\lambda,\ldots$ correspond once again to short-distance singularities which must be subtracted before multiplicative renormalisation. We are interested in the pole at $d=4$:

$$g_1(\boldsymbol{k})^{\text{sub}}=\frac{(N+2)u_0(2/\alpha)\pi f(\lambda,4)k^{2\lambda-\varepsilon}}{2^{2\lambda+1}\Gamma(1+\lambda)^2\lambda\sin\pi\lambda}\,\frac{1}{\varepsilon}. \tag{4.38}$$

Combining (4.34) and (4.38),

$$g(\boldsymbol{k})^{\text{sub}}=\frac{-(2/\alpha)\pi k^{2\lambda}}{2^{2\lambda+1}\Gamma(1+\lambda)^2\sin\pi\lambda}\left(1-\frac{(N+2)u_0f(\lambda,4)}{\lambda}\,\frac{k^{-\varepsilon}}{\varepsilon}+\ldots\right). \tag{4.39}$$

The renormalisation constant Z_2 is chosen, in minimal subtraction, so that $Z_2^2g(\boldsymbol{k})^{\text{sub}}$ should have no pole as $\varepsilon\to0$. Thus we take

$$Z_2=1+[(N+2)u_0f(\lambda,4)/2\lambda]\kappa^{-\varepsilon}/\varepsilon+\ldots \tag{4.40}$$

where κ is an arbitrary momentum scale. From (4.19) this leads to

$$d/2-1-\lambda-y_2(g)=[(N+2)f(\lambda,4)/2\lambda]g+\dots \tag{4.41}$$

$$=-(5\lambda^2+1)(N+2)\varepsilon/12\lambda(N+8)+O(\varepsilon^2) \tag{4.42}$$

at the fixed point. Using the scaling relation (4.25), this agrees with the result obtained for η_2 in § 3.

5. Summary and further remarks

Our main results are as follows. The singular part of the edge free energy per unit length scales according to

$$f_e^{(s)}=|t|^{(d-2)\nu}\psi(h|t|^{-y_0\nu}, h_1|t|^{-y_1\nu}, h_2|t|^{-y_2\nu}) \tag{5.1}$$

where t is the reduced temperature and h, h_1, h_2 are respectively bulk, surface and edge magnetic fields. The eigenvalues y_0 and y_1 are those previously studied in bulk and surface phenomena, and y_2 is a new exponent, given by

$$y_2=d/2-1-\lambda+(5\lambda^2+1)(N+2)\varepsilon/12\lambda(N+8)+O(\varepsilon^2) \tag{5.2}$$

where $\lambda=\pi/\alpha$. The spin correlations between spins in the bulk ($p=0$), near the surface ($p=1$), and near the edge ($p=2$), decay at criticality proportional to $r^{-(d-2+\eta_{p,q})}$, where

$$\eta_{p,q}=\tfrac{1}{2}(\eta_{p,p}+\eta_{q,q}) \tag{5.3}$$

and

$$\eta_{0,0}=\eta=d+2-2y_0 \tag{5.4}$$

$$\eta_{1,1}=\eta_{\parallel}=d-2y_1 \tag{5.5}$$

$$\eta_{2,2}=d-2-2y_2. \tag{5.6}$$

Magnetisation and susceptibility exponents follow from (5.1).

The main feature of our results is the dependence of the edge exponents on the angle α. In fact, they may depend on other features, such as the spatial anisotropy in the bulk. Our calculations so far have treated isotropic systems. If bulk correlations are anisotropic, it is necessary to rescale lengths anisotropically to bring the problem into an isotropic form. This will have the effect of changing α to an effective value dependent on the anisotropy. There will be no change if $\alpha=\pi$ (surface case), or if $\alpha=\pi/2$ and the edge is formed by the intersection of planes of mirror symmetry. This is illustrated in figure 2.

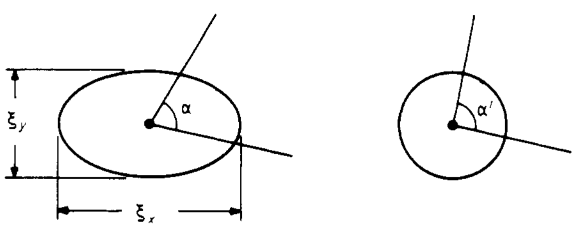

Figure 2. Anisotropy of bulk correlations is removed by a rescaling which changes α to α'.

The edge exponent β_2 could in principle be measured in a scattering experiment with momentum transfer parallel to the edge. Another possible application is to the statistics of self-avoiding walks in a wedge-shaped geometry, which corresponds to the limit $N \to 0$ of our results. Series calculations for the surface case have already been carried out for this problem (Barber *et al* 1978). It appears feasible to extend our calculations to $O(\varepsilon^2)$. The $N \to \infty$ limit, which is tractable for the surface problem (Bray and Moore 1977), appears to present more difficulties. We are at present carrying out real space RG calculations to test the qualitative features of our conclusions in lower dimensions.

In this paper we have considered what may be termed the ordinary edge transition, caused by the bulk going critical. We expect different edge behaviour at the surface transition, the extraordinary transition, and the special transition, which can occur for $d = 3$, $N = 1$. For $d \geq 4$ further transitions are conceivable where the edge orders before the surface or the bulk, but they will presumably not occur in $d = 3$ when the edge is one dimensional.

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